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ELASTIC FIELDS OF LINE DEFECTS IN A CRACKED BODY

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Abstract—Elastic fields are presented for line forces and dislocations in the vicinity of a crack tip and of a contained, double-ended planar crack. The fields of line force couples are also derived. The corresponding stress intensity factors are listed. The use of these results as two-dimensional Green functions for more general cases is discussed.

1. INTRODUCTION

Some representations of the elastic fields of line defects lying parallel to the tip of a planar crack have been available for a considerable period of time. Stroh[1] gave the methodology for determining such fields in the general anisotropic elastic case. This method has been used to determine the result for dislocations[2, 3] and for line forces[4] in anisotropic media. However, the results are presented in terms of roots of a sextic equation, the roots being the eigenvalues of the Stroh theory, and in general require numerical solutions including several matrix inversions. A somewhat simpler integral theory, which removes the need for the solution of the sextic equation, has been developed for line defects in continuous media[5, 7]. The integral theory cannot be applied to crack problems, though, because of the presence of fractional powers of the coordinates in the crack solutions which prevents a key simplification in choosing coordinates such that the coefficient of one of them is zero[5–7]. Thus, while the above results are useful for the case of a limited number of line defects near a crack, they become cumbersome for multiple defect problems and a simpler solution is desirable.

One can, of course, reduce the anisotropic results to the isotropic case. This procedure has the disadvantage of requiring cumbersome limiting procedures, however, as demonstrated explicity for the simplest, high-symmetry, anisotropic results for line defects in continuous media where analytical results are available[8, 9]. Therefore it is more straightforward to derive the isotropic results directly by the Muskhelishvili formalism[10, 11], the procedure followed in this work.

Some limited isotropic results are already in the literature. Rice and Thompson[12] gave the solution for the image force on a dislocation near a crack tip, while Hirth *et al.*[13] presented the stress function for line forces near a crack tip. Here, we present the generating function for the elastic fields of dislocations, line forces and line force doublets in a body containing a finite, double-ended crack and for the near-crack tip limit. Stress functions for these defects are also given.

The results can be used directly in the problem of current interest of describing the plastic zone near a crack tip in terms of continuous arrays of infinitesimal dislocations[14, 15]. Other direct applications include the use of the line forces, with their associated fields, in satisfying compatibility conditions at the boundary between a plastic and elastic zone in macroscopic problems, or between an atomistic (nonlinear elastic) and linear elastic zone in atomic calculations of defect fields by computer simulation[16, 17]. In addition the results may be used as two dimensional Green functions to derive the fields of other entities. For example, the field of a cylindrical inclusion can be determined by the integral about the inclusion-matrix interface of the field of a single line force directed normal to the inclusion surface.

In all cases, the results are presented in a form for which the reduction to real and imaginary parts is obvious. Hence, all results are presented in the briefer complex form.

2. THE CRACK SYSTEM

The geometry of the crack body is depicted in the complex plane in Figs. 1 and 2, respectively, for the single- and double-ended cracks. The arbitrary field positions relative to the crack ends are $z = r \exp i\theta$ and $z_0 = r_0 \exp i\theta_0$, while the coordinate positions of the pertinent line defect

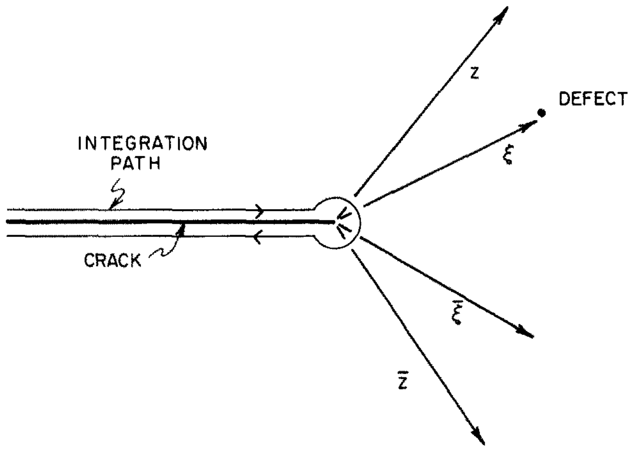

Fig. 1. Coordinate system in the complex plane for a cracked body. The crack tip is at the position of the origin and it and the line of the defect at ξ lie normal to the page.

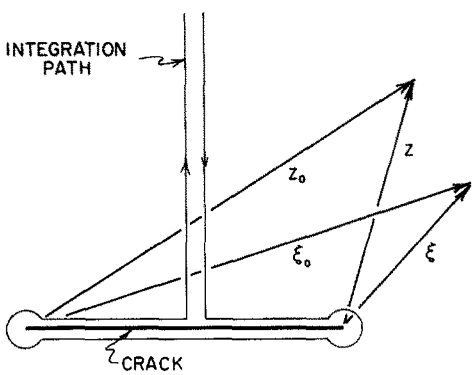

Fig. 2. Coordinate system in the complex plane for a body with a contained crack. The crack tips and the line of the defect at ξ lie normal to the page and the origin is at the center of the crack.

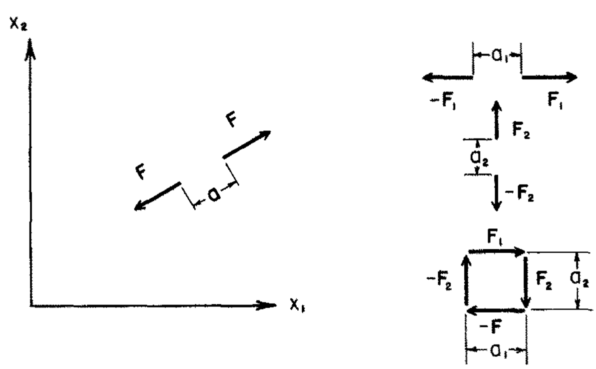

Fig. 3. Decomposition of a line force doublet without moment.

relative to the crack ends are $\xi = \rho \exp i\beta$ and $\xi_0 = \rho_0 \exp i\beta_0$.

Rice[11] has presented the appropriate line integrals for this type of problem for the stress intensity factors K_I, K_{II} and K_{III} and for the derivatives with respect to z, $\phi'(z)$ and $\omega'(z)$ of the functions $\phi(z)$ and $\omega(z)$ for plane strain (stress) and antiplane strain cases, respectively. In order to prevent possible confusion in signs, we present the corresponding contour integrals (Figs 1, 2). For the double-ended

$$\omega'(z) = \frac{1}{2\pi i(z^2 - L^2)^{1/2}} \oint \frac{p_3(t)(t^2 - L^2)^{1/2}\,dt}{t - z} \quad (1)$$

$$\phi'(z) = \frac{1}{4\pi i(z^2 - L^2)^{1/2}} \oint \frac{[p_2(t) - ip_1(t)](t^2 - L^2)^{1/2}\,dt}{t - z} \tag{2}$$

$$K_I - iK_{II} = \frac{1}{2(\pi L)^{1/2}} \oint \frac{[p_2(t) - ip_1(t)](t + L)^{1/2}\,dt}{(t - L)^{1/2}} \tag{3}$$

$$K_{III} = \frac{1}{2(\pi L)^{1/2}} \oint \frac{[p_3(t)](t + L)^{1/2}\,dt}{(t - L)^{1/2}} \tag{4}$$

crack where L is the half-crack length, $p_j(x_1) = -\sigma_{2j}(x_1, 0)$ and the latter stresses are those produced at the position of the crack by a defect in a medium without a crack. For the single-ended crack,

$$\omega'(z) = \frac{1}{2\pi i z^{1/2}} \oint \frac{p_3(t)t^{1/2}\,dt}{t - z} \tag{5}$$

$$\phi'(z) = \frac{1}{4\pi i z^{1/2}} \oint \frac{[p_2(t) - ip_1(t)]t^{1/2}\,dt}{t - z}. \tag{6}$$

In terms of these quantities, the stress intensity factors and displacements are

$$K_I - iK_{II} = \frac{1}{(2\pi)^{1/2} i} \oint \frac{[p_2(t) - ip_1(t)]\,dt}{\overline{(t)}^{1/2}} \tag{7}$$

$$K_{III} = \frac{1}{(2\pi)^{1/2} i} \oint \frac{p_3(t)\,dt}{\overline{(t)}^{1/2}} \tag{8}$$

$$u_1 + iu_2 = [\kappa\phi(z) - \phi(\bar{z}) - (z - \bar{z})\overline{\phi'(z)}]/2\mu \tag{9}$$

$$u_3 = \mathrm{Im}[\omega(z)/\mu] \tag{10}$$

where μ is the shear modulus, $\kappa = 3\text{–}4\nu$ for plane strain and $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress, with ν Poisson's ratio. The stresses are

$$\sigma_{22} - i\sigma_{12} = \phi'(z) + \phi'(\bar{z}) + (z - \bar{z})\overline{\phi''(z)} \tag{11}$$

$$\sigma_{11} = -\sigma_{22} + 2[\phi'(z) + \overline{\phi'(z)}]$$

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$$

$$\sigma_{23} + i\sigma_{13} = \omega'(z) \tag{12}$$

These forms for u_j and σ_{jk} are different from those presented by Rice[11] but can be derived from them with the use of his eqn (87).

3. SINGLE LINE DEFECT FIELDS

For line defects parallel to the x_3 axis of the corresponding cartesian coordinate system to those of Figs. 1 and 2, we consider arbitrary force vectors F_j and dislocation Burgers vectors b_j. For the dislocation, the sense vector points in the $+x_3$ direction and the convention is followed that b_j and the sense vector are coincident for a right-handed screw dislocation[8]. For the antiplane strain case the components b_3 and F_3 can be used directly. For the plane strain (stress) case, however, it is convenient to define the functions

$$F = (F_1 + iF_2)/2, \quad G = \mu b_1 + i\mu b_2 \tag{13}$$

and their complex conjugates $\bar{F}$, $\bar{G}$.

3.1 *Single-ended crack*

For the antiplane strain case, the generating function is

$$\omega(z) = -\frac{\mu b_3}{4\pi}[\phi_A(z) + \phi_B(z)] + \frac{iF_3}{4\pi}[\phi_A(z) - \phi_B(z)] \tag{14}$$

where

$$\phi_A(z) = \ln(z^{1/2} + \xi^{1/2})^2 \tag{15}$$
$$\phi_B(z) = \ln(z^{1/2} + \bar{\xi}^{1/2})^2.$$

For the in-plane case, the generating function is

$$\phi(z) = [(F + iG)\phi_A(z) + (-\kappa F + iG)\phi_B(z) + (\bar{F} - i\bar{G})\phi_C(z)]/2\pi(\kappa + 1). \tag{16}$$

The additional factor is

$$\phi_C(z) = (\xi - \bar{\xi})/\bar{\xi}^{1/2}(z^{1/2} + \bar{\xi}^{1/2}). \tag{17}$$

The function $\phi(\bar{z})$ is given by eqn (16) with z replaced by $\bar{z}$, but we note that

$$\phi_A(\bar{z}) = \overline{\phi_B(z)}, \quad \phi_B(\bar{z}) = \overline{\phi_A(z)} \tag{18}$$

of use in determining the stresses from eqn (11).

For the above functions, the derivatives to be used in eqns (11) and (12) can be readily obtained: for example, $\phi'_A(z) = 1/z^{1/2}(z^{1/2} + \xi^{1/2})$.

3.2 *Double-ended crack*

For the double-ended crack, the results, eqns (14) and (16) still apply but $\phi_A(z)$, $\phi_B(z)$ and $\phi_C(z)$ are changed. Rather than present the complete solution, we present the results in the form of terms to be added to those of eqns (15) and (17) to give the complete answer. The terms to be added are

$$\phi_A(z) = -2\ln[(z^{1/2} + \xi^{1/2})^2 - (z_0^{1/2} + \xi_0^{1/2})^2] + 2\ln(z_0^{1/2} + \xi_0^{1/2}) \tag{19}$$
$$\phi_B(z) = -2\ln[(z^{1/2} + \bar{\xi}^{1/2})^2 - (z_0^{1/2} + \bar{\xi}_0^{1/2})^2] + 2\ln(z_0^{1/2} + \bar{\xi}_0^{1/2})$$
$$\phi_C(z) = \frac{z^{1/2}(\xi - \bar{\xi})}{\bar{\xi}^{1/2}\bar{\xi}_0^{1/2}(z_0^{1/2} + \bar{\xi}_0^{1/2})}.$$

In this case the derivatives of eqn (19) for determining the stresses are not so simple. Hence, we present the contracted forms

$$\phi'_A(z) = \xi^{1/2}/z^{1/2}z_0^{1/2}(z_0^{1/2} + \xi_0^{1/2}) \tag{20}$$
$$\phi'_B(z) = \bar{\xi}^{1/2}/z^{1/2}z_0^{1/2}(z_0^{1/2} + \bar{\xi}_0^{1/2})$$
$$\phi'_C = \frac{(\xi - \bar{\xi})}{2\bar{\xi}^{1/2}z^{1/2}\bar{\xi}_0^{1/2}(z_0^{1/2} + \bar{\xi}_0^{1/2})} - \frac{(\xi - \bar{\xi})z^{1/2}}{2\bar{\xi}_0^{1/2}\bar{\xi}^{1/2}z_0^{1/2}(z_0^{1/2} + \bar{\xi}_0^{1/2})^2}.$$

3.3 *Stress intensity factors*

The stress intensity factors can be used in the usual manner, e.g. with eqns (78)–(83) in Rice[11], to give the near crack-tip stress fields. For the right-hand tip of the double-ended crack (the field at the other tip follows because of symmetry), with $|z| \ll |\xi|, |2L|$,

$$K_I - iK_{II} = \frac{1}{(\pi L)^{1/2}(\kappa + 1)}\left[(F + iG)\frac{\xi_0^{1/2}}{\xi^{1/2}} + (-\kappa F + iG)\frac{\bar{\xi}_0^{1/2}}{\bar{\xi}^{1/2}} - \frac{(\xi + \bar{j})L(\bar{F} - i\bar{G})}{\bar{\xi}_0^{1/2}\bar{\xi}^{3/2}}\right] \tag{21}$$

$$K_{III}=\frac{1}{4(\pi L)^{1/2}}\left[-\mu b_3\left(\frac{\xi_0^{1/2}}{\xi^{1/2}}+\frac{\bar{\xi}_0^{1/2}}{\bar{\xi}^{1/2}}\right)+iF_3\left(\frac{\xi_0^{1/2}}{\xi^{1/2}}-\frac{\bar{\xi}_0^{1/2}}{\bar{\xi}^{1/2}}\right)\right]. \tag{22}$$

For the single-ended crack, i.e. with $|z|\ll|\xi|\ll|2L|$,

$$K_I-iK_{II}=\frac{(2/\pi)^{1/2}}{(\kappa+1)}\left[\frac{F+iG}{\xi^{1/2}}+\frac{(-\kappa F+iG)}{\bar{\xi}^{1/2}}-\frac{(\xi-\bar{\xi})}{2\bar{\xi}^{3/2}}(\bar{F}-i\bar{G})\right] \tag{23}$$

$$K_{III}=\frac{1}{2(2\pi)^{1/2}}\left[-\mu b_3\left(\frac{1}{\xi^{1/2}}+\frac{1}{\bar{\xi}^{1/2}}\right)+iF_3\left(\frac{1}{\xi^{1/2}}-\frac{1}{\bar{\xi}^{1/2}}\right)\right]. \tag{24}$$

4. LINE FORCE DOUBLETS

For many defect fields, such as the dilatational field of a dislocation[19], it is convenient to directly use the fields of line force doublets as the pertinent Green function, since then force equilibrium is naturally satisfied. An arbitrary pair of line forces with force vectors $F_i(r)$ and $-F_i(r-a)$ separated by a vector a_j, defined conventionally in quadrant I to have positive components a_1 and a_2, with $a_3=0$, can be decomposed by superposition,[18], into a set of doublets M_{ij} as illustrated in Fig. 3. In M_{ij} the first index indicates the force direction and the second the component of a_j. The fields of the doublets are determined by letting the separation a_j approach zero while F_i approaches infinity in such a manner that the product $M_{ij}=F_ia_j$ remains constant. For any of the above functions, then,

$$f(M_{ij})=-\frac{\partial f(F_i)}{\partial x_j}a_j. \tag{25}$$

4.1 *Single-ended crack*

For the antiplane strain case the generating function is

$$\omega(z)=-\frac{(M_{32}-iM_{31})}{4\pi}\phi_D(z)-\frac{(M_{32}+iM_{31})}{4\pi}\phi_E(z) \tag{26}$$

where

$$\phi_D(z)=1/\xi^{1/2}(z^{1/2}+\xi^{1/2}),\quad \phi_E(z)=1/\bar{\xi}^{1/2}(z^{1/2}+\bar{\xi}^{1/2}). \tag{27}$$

For the plane strain (stress) case the generating function for the case of doublets without moment, i.e. $M_{12}=M_{21}$, is

$$\phi(z)=\frac{1}{4\pi(\kappa+1)}[M_{11}-M_{22}+2iM_{12}]\phi_D(z)+[-\kappa M_{11}-(\kappa-2)M_{22}+2iM_{12}]\phi_E(z)$$
$$-[(\xi-\bar{\xi})/2][M_{11}-M_{22}-2iM_{12}]\phi_F(z) \tag{28}$$

where

$$\phi_F(z)=[1/\bar{\xi}^{3/2}(z^{1/2}+\bar{\xi}^{1/2})]+[1/\bar{\xi}(z^{1/2}+\bar{\xi}^{1/2})^2] \tag{29}$$

For the case of doublets with moment, $M_{12}\neq M_{21}$, the function to be added to eqn (28) is

$$\phi(z)=\frac{1}{4\pi(\kappa+1)}\{i(M_{21}-M_{12})\phi_D(z)+i\kappa(M_{12}-M_{21})\phi_E(z)$$
$$+[(\xi-\bar{\xi})/2][iM_{21}-iM_{12}]\phi_F(z)\}. \tag{30}$$

4.2 *Double-ended crack*

Equation (26) also gives the result for the antiplane strain case for the double-ended crack, but eqn (27) is replaced by

$$\phi_D(z)=z^{1/2}/\xi^{1/2}\xi_0^{1/2}(z_0^{1/2}+\xi_0^{1/2});\quad \phi_E(z)=z^{1/2}/\bar{\xi}^{1/2}\bar{\xi}_0^{1/2}(z_0^{1/2}+\bar{\xi}_0^{1/2}). \tag{31}$$

For the plane strain (stress) case, eqns (28) and (30) apply but eqn (27) is replaced by (31) and eqn (29) is replaced by

$$\phi_F(z) = \frac{z^{1/2}(\bar{\xi} + \bar{\xi}_0)}{\bar{\xi}^{3/2}\bar{\xi}_0^{3/2}(z_0^{1/2} + \bar{\xi}_0^{1/2})} + \frac{z^{1/2}}{2\bar{\xi}^{1/2}\bar{\xi}_0(z_0^{1/2} + \bar{\xi}_0^{1/2})^2}. \tag{32}$$

4.3 *Stress intensity factors*

For the double-ended crack, the stress intensity factors for the force doublets, which give the stresses when $|z| \ll |\xi|$, $|2L|$ as discussed in the previous section are, for the case $M_{12} = M_{21}$,

$$\begin{aligned} K_I - iK_{II} = -\frac{(\pi L)^{1/2}}{2\pi(\kappa + 1)} &\{[M_{11} - M_{22} + 2iM_{12}]/\xi_0^{1/2}\xi^{3/2} \\ &+ [-\kappa M_{11} - (\kappa - 2)M_{22} + 2iM_{12}]/\bar{\xi}_0^{1/2}\bar{\xi}^{3/2} \\ &- [(\xi - \bar{\xi})[M_{11} - M_{22} - 2iM_{12}]](3/2\bar{\xi}_0^{1/2}\bar{\xi}^{5/2}) + (1/2\bar{\xi}_0^{3/2}\bar{\xi}^{3/2})]\}. \end{aligned} \tag{33}$$

The added terms for the case $M_{12} \neq M_{21}$ are

$$\begin{aligned} K_I - iK_{II} = \frac{(\pi L)^{1/2}}{2\pi(\kappa + 1)} &\left\{ \frac{i(M_{12} - M_{21})}{\xi_0^{1/2}\xi^{3/2}} + \frac{i(M_{21} - M_{12})}{\bar{\xi}_0^{1/2}\bar{\xi}^{3/2}} + i(\xi - \bar{\xi})(M_{12} - M_{21}) \right. \\ &\left. \times [(3/2\bar{\xi}_0^{1/2}\bar{\xi}^{5/2}) + (1/2\bar{\xi}_0^{3/2}\bar{\xi}^{3/2})] \right\}. \end{aligned} \tag{34}$$

For the antiplane strain case

$$K_{III} = \frac{(\pi L)^{1/2}}{4\pi} \left\{ \frac{(M_{32} - iM_{31})}{\xi_0^{1/2}\xi^{3/2}} + \frac{(M_{32} + iM_{31})}{\bar{\xi}_0^{1/2}\bar{\xi}^{3/2}} \right\}. \tag{35}$$

For the single-ended crack, i.e. when $|z| \ll |\xi| \ll 2L$, the stress intensity factors for the doublets are, for the case $M_{12} = M_{21}$,

$$\begin{aligned} K_I - iK_{II} = \frac{-(2\pi)^{1/2}}{4\pi(\kappa + 1)} &\{[M_{11} - M_{22} + 2iM_{12}]/\xi^{3/2} + [-\kappa M_{11} - (\kappa - 2)M_{22} + 2iM_{12}]/\bar{\xi}^{3/2} \\ &- 3[M_{11} - M_{22} - 2iM_{12}](\xi - \bar{\xi})/2\bar{\xi}^{5/2}\}. \end{aligned} \tag{36}$$

The added terms for the case $M_{12} \neq M_{21}$ are

$$K_I - iK_{II} = \frac{-(2\pi)^{1/2}}{4\pi(\kappa + 1)} \left\{ \frac{i(M_{21} - M_{12})}{\xi^{3/2}} + \frac{i\kappa(M_{12} - M_{21})}{\bar{\xi}^{3/2}} + \frac{3i(M_{21} - M_{12})(\xi - \bar{\xi})}{2\bar{\xi}^{5/2}} \right\}. \tag{37}$$

Finally, for the antiplane strain case,

$$K_{III} = \frac{(2\pi)^{1/2}}{8\pi} \left\{ \frac{(M_{32} - iM_{31})}{\xi^{3/2}} + \frac{(M_{32} + iM_{31})}{\bar{\xi}^{3/2}} \right\}. \tag{38}$$

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